Principles of Fractional Quantum Mechanics

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Abstract

A review of fundamentals and physical applications of fractional quantum mechanics has been presented.

Fundamentals cover fractional Schrödinger equation, quantum Riesz fractional derivative, path integral approach to fractional quantum mechanics, hermiticity of the Hamilton operator, parity conservation law and the current density. Applications of fractional quantum mechanics cover dynamics of a free particle, new representation for a free particle quantum mechanical kernel, infinite potential well, bound state in δ-potential well, linear potential, fractional Bohr atom and fractional oscillator.

We also review fundamentals of the Lévy path integral approach to fractional statistical mechanics.

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1 Introduction

Classical mechanics and quantum mechanics are based on the assumption that the Hamilton function has the form

$$H(\mathbf{p},\mathbf{r}) = \frac{\mathbf{p}^2}{2m} + V(\mathbf{r}), \tag{1}$$

where $\mathbf{p}$ and $\mathbf{r}$ are the momentum and space coordinate of a particle with mass m, and $V(\mathbf{r})$ is the potential energy. In quantum mechanics, $\mathbf{p}$ and $\mathbf{r}$ should be considered as quantum mechanical operators $\widehat{\mathbf{p}}$ and $\widehat{\mathbf{r}}$. Then the Hamiltonian function $H(\mathbf{p},\mathbf{r})$ becomes the Hamilton operator $\widehat{H}(\widehat{\mathbf{p}},\widehat{\mathbf{r}})$,

$$\widehat{H}(\widehat{\mathbf{p}},\widehat{\mathbf{r}}) = \frac{\widehat{\mathbf{p}}^2}{2m} + \widehat{V}(\widehat{\mathbf{r}}), \tag{2}$$

where $\widehat{V}(\widehat{\mathbf{r}})$ is the potential energy operator.

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The square dependence on the momentum in Eqs.(1) and (2) is empirical physical fact. However an attempt to get insight on the fundamentals behind this fact posts the question: are there other forms of kinematic term in Eqs.(1) and (2) which do not contradict the fundamental principles of classical mechanics and quantum mechanics? A convenient theoretical physics approach to answer this question is the Feynman path integral approach to quantum mechanics [1], as it was first observed by Laskin [2]. Indeed, the Feynman path integral is the integral over Brownian-like paths. Brownian motion is a special case of so-called α-stable probability distributions developed by Lévy [3] and Khintchine [4]. In mid 1930's they posed the question: Does the sum of N independent identically distributed random quantities $X = X_1 + X_2 ... + X_N$ have the same probability distribution $p_N(X)$ (up to scale factor) as the individual steps $p_i(X_i)$, $i = 1, ...N$? The traditional answer is that each $p_i(X_i)$ should be a Gaussian, because of the central limit theorem. In other words, a sum of N Gaussians is again a Gaussian, but with N times the variance of the original. Lévy and Khintchine proved that there exist the possibility to generalize the central limit theorem. They discovered a class of non-Gaussian α-stable (stable under summation) probability distributions. Each α-stable distribution has a stability index α, often called the Lévy index $0 < \alpha \leq 2$. When $\alpha = 2$ Lévy motion is transformed into Brownian motion.

An option to develop the path integral over Lévy paths was discussed by Kac [5], who pointed out that the Lévy path integral generates a functional measure in the space of left (or right) continuos functions (paths) having only discontinuities of the first kind. The path integral over the Lévy paths has first been introduced and elaborated with applications to fractional quantum mechanics and fractional statistical mechanics by Laskin (see, [2], [14]-[17]). He followed the framework of the Feynman space-time vision of quantum mechanics, but instead of the Brownian-like quantum mechanical trajectories, Laskin used the Lévy-like ones. If the fractal dimension (for definition of fractal dimension, see [6], [7]) of the Brownian path is $\mathrm{d}_{\mathrm{fractal}}^{(Brownian)}$ =2, then the Lévy path has fractal dimension $\mathrm{d}_{\mathrm{fractal}}^{(Lévy)} = \alpha$, where α is so-called the Lévy index, $1 < \alpha \leq 2$. The Lévy index α becomes a new fundamental parameter in fractional quantum and classical mechanics similar to $\mathrm{d}_{\mathrm{fractal}}^{(Brownian)} = 2$ being a fundamental parameter in standard quantum and classical mechanics. The difference between the fractal dimensions of the Brownian and Lévy paths leads to different physics. In fact, fractional quantum dynamics is generated by the Hamiltonian function $H_\alpha(\mathbf{p}, \mathbf{r})$ of the form [2], [14]-[17]

$$H_\alpha(\mathbf{p}, \mathbf{r}) = D_\alpha |\mathbf{p}|^\alpha + V(\mathbf{r}), \qquad 1 < \alpha \leq 2, \tag{3}$$

with substitutions $\mathbf{p} \rightarrow \widehat{\mathbf{p}}$, $\mathbf{r} \rightarrow \widehat{\mathbf{r}}$, and D_α is the generalized coefficient, the physical dimension of which is $[D_\alpha] = \mathrm{erg}^{1-\alpha} \cdot \mathrm{cm}^\alpha \cdot \mathrm{sec}^{-\alpha}$. One can say that Eq.(3) is a natural generalization of the well-known Eq.(1). When $\alpha = 2$, $D_\alpha = 1/2m$ and Eq.(3) is transformed into Eq.(1) [2]. As a result, the fractional quantum mechanics based on the Lévy path integral generalizes the standard quantum mechanics based on the well-known Feynman path integral. Indeed, if

the path integral over Brownian trajectories leads to the well-known Schrödinger equation, then the path integral over Lévy trajectories leads to the fractional Schrödinger equation. The fractional Schrödinger equation is a new fundamental equation of quantum physics and it includes the space derivative of order α instead of the second ($\alpha = 2$) order space derivative in the standard Schrödinger equation. Thus, the fractional Schrödinger equation is the fractional differential equation in accordance with modern terminology (see, for example, [9]-[13]). This is the main point for the term *fractional Schrödinger equation* or for more general term *fractional quantum mechanics*, FQM [2], [14]. When Lévy index $\alpha = 2$, Lévy motion becomes Brownian motion. Thus, FQM includes standard QM as a particular Gaussian case at $\alpha = 2$. The quantum mechanical path integral over the Lévy paths [2] at $\alpha = 2$ becomes the Feynman path integral [1].

In the limit case $\alpha = 2$ the fundamental equations of fractional quantum mechanics are transformed into the well-known equations of standard quantum mechanics [1], [8], [24].

2 Fractional Schrödinger equation

2.1 Quantum Riesz fractional derivative

Equation (1) lets us conclude that the energy E of a particle of mass m under the influence of the potential $V(\mathbf{r})$ is given by

$$E = \frac{\mathbf{p}^2}{2m} + V(\mathbf{r}). \tag{4}$$

To obtain the Schrödinger equation we introduce the operators following the well-known procedure,

$$E \to i\hbar\frac{\partial}{\partial t}, \qquad \mathbf{p} \to -i\hbar\boldsymbol{\nabla}, \tag{5}$$

where $\boldsymbol{\nabla} = \partial/\partial\mathbf{r}$ and $\hbar$ is the Planck's constant. Further, substituting transformation (5) into Eq.(1) and applying them to the wave function $\psi(\mathbf{r}, t)$ yields

$$i\hbar\frac{\partial\psi(\mathbf{r},t)}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi(\mathbf{r},t) + V(\mathbf{r})\psi(\mathbf{r},t), \tag{6}$$

here $\Delta = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}$ is the Laplacian. Thus, we obtain the Schrödinger equation [8].

By repeating the same consideration to Eq.(3) we find the fractional Schrödinger equation [2], [16]

$$i\hbar\frac{\partial\psi(\mathbf{r},t)}{\partial t} = D_\alpha(-\hbar^2\Delta)^{\alpha/2}\psi(\mathbf{r},t) + V(\mathbf{r})\psi(\mathbf{r},t), \qquad 1 < \alpha \leq 2, \tag{7}$$

with 3D generalization of the fractional quantum Riesz derivative $(-\hbar^2\Delta)^{\alpha/2}$ introduced by

$$(-\hbar^2\Delta)^{\alpha/2}\psi(\mathbf{r},t) = \frac{1}{(2\pi\hbar)^3}\int d^3p e^{i\frac{\mathbf{pr}}{\hbar}}|\mathbf{p}|^\alpha\varphi(\mathbf{p},t), \tag{8}$$

where the wave functions in space $\psi(\mathbf{r},t)$ and momentum $\varphi(\mathbf{p},t)$ representations are related each to other by the 3D Fourier transforms

$$\psi(\mathbf{r},t) = \frac{1}{(2\pi\hbar)^3}\int d^3p e^{i\frac{\mathbf{pr}}{\hbar}}\varphi(\mathbf{p},t), \qquad \varphi(\mathbf{p},t) = \int d^3r e^{-i\frac{\mathbf{pr}}{\hbar}}\psi(\mathbf{r},t). \tag{9}$$

The 3D fractional Schrödinger equation Eq.(7) has the following operator form

$$i\hbar\frac{\partial\psi(\mathbf{r},t)}{\partial t} = \widehat{H}_\alpha(\widehat{\mathbf{p}},\widehat{\mathbf{r}})\psi(\mathbf{r},t),$$

where fractional Hamilton operator $\widehat{H}_\alpha(\widehat{\mathbf{p}},\widehat{\mathbf{r}})$ results from Eq.(3) with quantum-mechanical operators $\widehat{\mathbf{p}}$ and $\widehat{\mathbf{r}}$ substituted instead of $\mathbf{p}$ and $\mathbf{r}$,

$$\widehat{H}_\alpha(\widehat{\mathbf{p}},\widehat{\mathbf{r}}) = D_\alpha|\widehat{\mathbf{p}}|^\alpha + V(\widehat{\mathbf{r}}), \qquad 1 < \alpha \le 2. \tag{10}$$

The 1D fractional Schrödinger equation has the form [2], [14]-[16]

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = -D_\alpha(\hbar\nabla)^\alpha\psi(x,t) + V(x)\psi(x,t), \qquad 1 < \alpha \le 2, \tag{11}$$

where $(\hbar\nabla)^\alpha$ is the quantum Riesz fractional derivative[1]. The quantum Riesz fractional derivative is defined by the following way [2], [14]

$$(\hbar\nabla)^\alpha\psi(x,t) = -\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dp e^{i\frac{px}{\hbar}}|p|^\alpha\varphi(p,t), \tag{12}$$

where $\varphi(p,t)$ is the Fourier transform of the wave function $\psi(x,t)$ given by

$$\varphi(p,t) = \int_{-\infty}^{\infty} dx e^{-i\frac{px}{\hbar}}\psi(x,t), \tag{13}$$

and reciprocally

$$\psi(x,t) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dp e^{i\frac{px}{\hbar}}\varphi(p,t).$$

It is easy to see that Eq.(11) can be rewritten in the operator form, namely

[1] The Riesz fractional derivative was originally introduced in [23]

$$i\hbar\frac{\partial\psi}{\partial t} = H_\alpha\psi, \tag{14}$$

where H_α is the fractional Hamilton operator

$$H_\alpha = -D_\alpha(\hbar\nabla)^\alpha + V(x). \tag{15}$$

For the special case when $\alpha = 2$ and $D_2 = 1/2m$ (see, for details [2], [14]), where m is the particle mass, Eqs.(7) and (11) are transformed into the well-known 3D and 1D Schrödinger equations [8].

2.2 The hermiticity of the fractional Hamilton operator

The fractional Hamiltonian H_α given by Eq.(15) is the Hermitian operator in the space with scalar product

$$(\phi, \chi) = \int_{-\infty}^{\infty} dx\phi^*(x,t)\chi(x,t). \tag{16}$$

To prove the hermiticity of H_α let us note that in accordance with the definition of the quantum Riesz fractional derivative given by Eq.(12) there exists the integration-by parts formula

$$(\phi, (\hbar\nabla)^\alpha\chi) = ((\hbar\nabla)^\alpha\phi, \chi). \tag{17}$$

The average energy of a fractional quantum system with Hamiltonian H_α is

$$E_\alpha = \int_{-\infty}^{\infty} dx\psi^*(x,t)H_\alpha\psi(x,t). \tag{18}$$

Taking into account Eq.(17) we have

$$E_\alpha = \int_{-\infty}^{\infty} dx\psi^*(x,t)H_\alpha\psi(x,t) = \int_{-\infty}^{\infty} dx(H_\alpha^+\psi(x,t))^*\psi(x,t) = E_\alpha^*.$$

As a physical consequence, the energy of the system is real. Thus, the fractional Hamiltonian H_α defined by Eq.(15) is the Hermitian or self-adjoint operator in the space with the scalar product defined by Eq.(16) [15], [16]

$$(H_\alpha^+\phi, \chi) = (\phi, H_\alpha\chi). \tag{19}$$

The generalization of the proof of hermiticity for 3D case is straightforward. Note that Eq.(11) leads to the important equation

$$\frac{\partial}{\partial t}\int dx\psi^*(x,t)\psi(x,t) = 0, \tag{20}$$

which shows that the wave function remains normalized, if it is normalized once. Multiplying Eq.(11) from the left by $\psi^*(x,t)$ and the conjugate complex of Eq.(11) by $\psi(x,t)$ and then subtracting the two resultant equations finally yield

$$i\hbar\frac{\partial}{\partial t}\left(\psi^*(x,t)\psi(x,t)\right) = \psi^*(x,t)H_\alpha\psi(x,t) - \psi(x,t)H_\alpha^*\psi^*(x,t).$$

Integrating this relation over all values of the space variable and using the fact that the operator H_α is self-adjoint, we find Eq.(20). The above consideration can be easily generalized to 3D case.

2.3 The parity conservation law for the fractional quantum mechanics

It follows from the definition (12) of the quantum Riesz fractional derivative that

$$(\hbar\nabla)^\alpha \exp\{i\frac{px}{\hbar}\} = -|p|^\alpha \exp\{i\frac{px}{\hbar}\}, \tag{21}$$

in other words, the function $\exp\{ipx/\hbar\}$ is the eigenfunction of the quantum Riesz fractional operator $(\hbar\nabla)^\alpha$ with eigenvalue $-|p|^\alpha$.

The 3D generalization is straightforward,

$$(-\hbar^2\Delta)^{\alpha/2} \exp\{i\frac{\mathbf{px}}{\hbar}\} = |\mathbf{p}|^\alpha \exp\{i\frac{\mathbf{px}}{\hbar}\}, \tag{22}$$

the function $\exp\{i\mathbf{px}/\hbar\}$ is the eigenfunction of 3D quantum Riesz fractional operator $(-\hbar^2\Delta)^{\alpha/2}$ with eigenvalue $|\mathbf{p}|^\alpha$.

Thus, the operators $(\hbar\nabla)^\alpha$ and $(-\hbar^2\Delta)^{\alpha/2}$ are the symmetrized fractional derivative, that is

$$(\hbar\nabla_x)^\alpha ... = (\hbar\nabla_{-x})^\alpha ..., \tag{23}$$

$$(-\hbar^2\Delta_{\mathbf{r}})^{\alpha/2} ... = (-\hbar^2\Delta_{-\mathbf{r}})^{\alpha/2} .... \tag{24}$$

Because of the properties (17) and (18) the fractional Hamiltonian H_α (see, for example Eqs.(5) or (8)) remains invariant under *inversion* transformation. Inversion, or to be precise, spatial inversion consists in the simultaneous change in sign of all three spatial coordinates

$$\mathbf{r} \to -\mathbf{r}, \qquad x \to -x, \quad y \to -y, \quad z \to -z. \tag{25}$$

Let us denote the inversion operator by $\widehat{P}$. The inverse symmetry is the fact that $\widehat{P}$ and the fractional Hamiltonian H_α commute,

$$\widehat{P}H_\alpha = H_\alpha\widehat{P}. \tag{26}$$

We can divide the wave functions of quantum mechanical states with a well-defined eigenvalue of the operator $\widehat{P}$ into two classes; (i) functions which are

not changed when acted upon by the inversion operator, $\widehat{P}\psi_+(\mathbf{r}) = \psi_+(\mathbf{r})$ the corresponding states are called even states; (ii) functions which change sign under the action of the inversion operator, $\widehat{P}\psi_-(\mathbf{r}) = -\psi_-(\mathbf{r})$ the corresponding states are called odd states. Eq.(26) express the "parity conservation law" for the FQM [16]; if the state of a closed fractional quantum mechanical system has a given parity (i.e. if it is even, or odd), then this parity is conserved.

2.4 The current density

By multiplying Eq.(7) from left by $\psi^*(\mathbf{r},t)$ and the conjugate complex of Eq.(7) by $\psi(\mathbf{r},t)$ and subtracting the two resultant equations yield

$$\frac{\partial}{\partial t}\int d^3r\,(\psi^*(\mathbf{r},t)\psi(\mathbf{r},t)) = \tag{27}$$

$$\frac{D_\alpha}{i\hbar}\int d^3r\left(\psi^*(\mathbf{r},t)(-\hbar^2\Delta)^{\alpha/2}\psi(\mathbf{r},t) - \psi(\mathbf{r},t)(-\hbar^2\Delta)^{\alpha/2}\psi^*(\mathbf{r},t)\right).$$

From this integral relationship we are led to the following well-known differential equation

$$\frac{\partial\rho(\mathbf{r},t)}{\partial t} + \mathrm{div}\mathbf{j}(\mathbf{r},t) = 0, \tag{28}$$

where $\rho(\mathbf{r},t) = \psi^*(\mathbf{r},t)\psi(\mathbf{r},t)$ is the quantum mechanical probability density and the vector $\mathbf{j}(\mathbf{r},t)$ can be called by the fractional probability current density vector

$$\mathbf{j}(\mathbf{r},t) = \frac{D_\alpha\hbar}{i}\left(\psi^*(\mathbf{r},t)(-\hbar^2\Delta)^{\alpha/2-1}\boldsymbol{\nabla}\psi(\mathbf{r},t) - \psi(\mathbf{r},t)(-\hbar^2\Delta)^{\alpha/2-1}\boldsymbol{\nabla}\psi^*(\mathbf{r},t)\right), \tag{29}$$

where we use the following notation $\boldsymbol{\nabla} = \partial/\partial\mathbf{r}$. Introducing the momentum operator $\widehat{\mathbf{p}} = \frac{\hbar}{i}\frac{\partial}{\partial\mathbf{r}}$ we can write the vector $\mathbf{j}$ in the form [16]

$$\mathbf{j} = D_\alpha\left(\psi(\widehat{\mathbf{p}}^2)^{\alpha/2-1}\widehat{\mathbf{p}}\psi^* + \psi^*(\widehat{\mathbf{p}}^{*2})^{\alpha/2-1}\widehat{\mathbf{p}}^*\psi\right) \tag{30}$$

The new fundamental Eqs.(29) and (30) are the fractional generalization of the well-known equations for probability current density vector of standard quantum mechanics [8].

To this end, we express Eq.(30) in the terms of the velocity operator, which is defined as follows $\widehat{\mathbf{v}} = d\widehat{\mathbf{r}}/dt$, where $\widehat{\mathbf{r}}$ is the operator of coordinate. Using the general quantum mechanical rule for differentiation of operator

$$\frac{d}{dt}\widehat{\mathbf{r}} = \frac{i}{\hbar}[H_\alpha, \mathbf{r}],$$

we have

$$\widehat{\mathbf{v}} = \frac{i}{\hbar}(H_\alpha \mathbf{r} - \mathbf{r} H_\alpha),$$

Further, with help of the equation $\mathrm{f}(\widehat{\mathbf{p}})\mathbf{r} - \mathbf{r}\mathrm{f}(\widehat{\mathbf{p}}) = -i\hbar\partial \mathrm{f}/\partial\mathbf{p}$, which holds for any function $\mathrm{f}(\widehat{\mathbf{p}})$ of the momentum operator, and taking into account Eq.(10) for the Hamiltonian operator $\widehat{H}_\alpha(\widehat{\mathbf{p}}, \widehat{\mathbf{r}})$ we obtain the equation for the velocity operator

$$\widehat{\mathbf{v}} = \alpha D_\alpha |\widehat{\mathbf{p}}^2|^{\alpha/2-1}\widehat{\mathbf{p}}, \tag{31}$$

here $\widehat{\mathbf{p}}$ is the momentum operator. By comparing of Eqs.(30) and (31) we finally conclude that

$$\mathbf{j} = \frac{1}{\alpha}\left(\psi\widehat{\mathbf{v}}\psi^* + \psi^*\widehat{\mathbf{v}}\psi\right), \qquad 1 < \alpha \leq 2. \tag{32}$$

To get the probability current density equal to 1 (the current when one particle passes through unit area per unit time) the wave function of a free particle has to be normalized as

$$\psi(\mathbf{r}, t) = \sqrt{\frac{\alpha}{2\mathrm{v}}}\exp\{\frac{i}{\hbar}\mathbf{pr} - \frac{i}{\hbar}Et\}, \qquad E = D_\alpha|\mathbf{p}|^\alpha, \qquad 1 < \alpha \leq 2, \tag{33}$$

where v is the particle velocity, $\mathrm{v} = \alpha D_\alpha p^{\alpha-1}$. Then we have

$$\mathbf{j} = \frac{\mathbf{v}}{\mathrm{v}}, \qquad \mathbf{v} = \alpha D_\alpha |\mathbf{p}^2|^{\frac{\alpha}{2}-1}\mathbf{p}, \tag{34}$$

that is, the vector $\mathbf{j}$ is indeed the unit vector.

Equations (29)-(34) are the fractional generalization of the well-known equations for probability current density vector and velocity vector of the standard quantum mechanics [8].

2.5 The time-independent fractional Schrödinger equation

The special case when the Hamiltonian H_α does not depend explicitly on the time is of great importance for physical applications. It is easy to see that in this case there exist the special solution of the fractional Schrödinger equation (11) of the form

$$\psi(x, t) = e^{-(i/\hbar)Et}\phi(x), \tag{35}$$

where $\phi(x)$ satisfies

$$H_\alpha\phi(x) = E\phi(x), \tag{36}$$

or

$$-D_\alpha(\hbar\nabla)^\alpha\phi(x) + V(x)\phi(x) = E\phi(x), \qquad 1 < \alpha \leq 2. \tag{37}$$

The equation (37) we call by the time-independent (or stationary) fractional Schrödinger equation [15], [16]. We see from Eq.(35) that the wave function $\psi(x,t)$ oscillates with a definite frequency. The frequency with which a wave function oscillates corresponds to the energy. Therefore, we say that when the fractional wave function $\psi(x,t)$ is of this special form, the state has a definite energy E. The probability to find a particle at x is the absolute square of the wave function $|\psi|^2$. In view of Eq.(35) this is equal to $|\phi|^2$and does not depend upon the time. That is, the probability of finding the particle in any location is independent of the time. We say under these circumstances that the system is in a stationary state - stationary in the sense that there is no variation in the probabilities as a function of time.

3 Path integral

If a particle at an initial time t_a starts from the point x_a and goes to a final point x_b at time t_b, we will say simply that the particle goes from a to b and its trajectory (path)[2] $x(t)$ will have the property that $x(t_a) = x_a$ and $x(t_b) = x_b$. In quantum mechanics, then, we will have an quantum-mechanical amplitude, often called a kernel, which we may write $K(x_b t_b|x_a t_a)$, to get from the point a to the point b. This will be the sum over all of the trajectories that go between that end points and of a contribution from each [1]. For the one dimensional version of the Hamiltonian Eq.(3)

$$H_\alpha(p,x) = D_\alpha |p|^\alpha + V(x,t), \tag{38}$$

following consideration provided in [2], [14], we come to the definition of the kernel $K(x_b t_b|x_a t_a)$ in terms of path integral in the phase space representation

$$K(x_b t_b|x_a t_a) = \lim_{N\to\infty} \int\limits_{-\infty}^{\infty} dx_1 ... dx_{N-1} \frac{1}{(2\pi\hbar)^N} \int\limits_{-\infty}^{\infty} dp_1 ... dp_N \times \tag{39}$$

$$\exp\left\{ \frac{i}{\hbar} \sum_{j=1}^{N} p_j (x_j - x_{j-1}) \right\} \times \exp\left\{ -\frac{i}{\hbar} D_\alpha \varepsilon \sum_{j=1}^{N} |p_j|^\alpha - \frac{i}{\hbar} \varepsilon \sum_{j=1}^{N} V(x_j, j\varepsilon) \right\},$$

here $\varepsilon = (t_b - t_a)/N$, $x_j = x(j\varepsilon)$, $p_j = p(j\varepsilon)$ and $x_0 = x_a$, $x_N = x_b$. Then in the continuum limit $N \to \infty, \quad \varepsilon \to 0$ we have

$$K_L(x_b t_b|x_a t_a) = \tag{40}$$

$$\int\limits_{x(t_a)=x_a}^{x(t_b)=x_b} \mathrm{D}x(\tau) \int \mathrm{D}p(\tau) \exp\left\{ \frac{i}{\hbar} \int\limits_{t_a}^{t_b} d\tau [p(\tau)\dot{x}(\tau) - H_\alpha(p(\tau), x(\tau), \tau)] \right\},$$

[2]For simplicity, here we consider one dimensional motion.

where $\dot{x}$ denotes the time derivative, $H_\alpha(p(\tau), x(\tau), \tau)$ is the fractional Hamiltonian given by Eq.(38) with the replacement $p \to p(\tau)$, $x \to x(\tau)$ and $\{p(\tau), x(\tau)\}$ is the particle trajectory in phase space, and, finally, the phase space path integral $\int\limits_{x(t_a)=x_a}^{x(t_b)=x_b} \mathrm{D}x(\tau) \int \mathrm{D}p(\tau)...$ is given by

$$\int\limits_{x(t_a)=x_a}^{x(t_b)=x_b} \mathrm{D}x(\tau) \int \mathrm{D}p(\tau)... = \tag{41}$$

$$= \lim_{N\to\infty} \int\limits_{-\infty}^{\infty} dx_1...dx_{N-1} \frac{1}{(2\pi\hbar)^N} \int\limits_{-\infty}^{\infty} dp_1...dp_N \times$$

$$\exp\left\{i\frac{p_1(x_1 - x_a)}{\hbar} - i\frac{D_\alpha\varepsilon|p_1|^\alpha}{\hbar}\right\}\times...\times\exp\left\{i\frac{p_N(x_b - x_{N-1})}{\hbar} - i\frac{D_\alpha\varepsilon|p_N|^\alpha}{\hbar}\right\}...,$$

The exponential in Eq.(40) can be written as $\exp\{\frac{i}{\hbar}S_\alpha(p,x)\}$ if we introduce the fractional canonical classical mechanical action $S_\alpha(p,x)$ for the trajectory $p(t)$, $x(t)$ in phase space [14]

$$S_\alpha(p,x) = \int\limits_{t_a}^{t_b} d\tau(p(\tau)\dot{x}(\tau) - H_\alpha(p(\tau), x(\tau), \tau). \tag{42}$$

Since the coordinates x_0, x_N in the definition (41) are fixed at their initial and final points, $x_0 = x_a$ and $x_N = x_b$, the all possible trajectories in Eq.(40) satisfy the boundary condition $x(t_b) = x_b$, $x(t_a) = x_a$. We see that the definition given by Eq.(41) includes one more p_j-integrals than x_j-integrals. Indeed, while x_0 and x_N are held fixed and the x_j-integrals are done for $j = 1, ..., N-1$, each increment $x_j - x_{j-1}$ is accompanied by one p_j-integral for $j = 1, ..., N$. The above observed asymmetry is a consequence of the particular boundary condition. Namely, the end points are fixed in the position (coordinate) space. There exist the possibility of proceeding in a conjugate way keeping the initial p_a and final p_b momenta and fixed. The associated kernel can be derived going through the same steps as before but working in the momentum representation (see, for example, [24]).

The kernel $K(x_bt_b|x_at_a)$ introduced by Eq.(40) describes the evolution of the quantum mechanical system

$$\psi(x_b, t_b) = \int\limits_{-\infty}^{\infty} dx_a K(x_bt_b|x_at_a)\psi(x_a, t_a), \tag{43}$$

where $\psi(x_a, t_a)$ is the wave function of the initial state (at $t = t_a$) and $\psi(x_b, t_b)$ is the wave function of the final state (at $t = t_b$).

3.1 Free particle

For a free particle when $V(x,t)$, we have $H_\alpha(p) = D_\alpha|p|^\alpha$ and it's easy to see that Eq.(39) results in [2]

$$K^{(0)}(x_b t_b|x_a t_a) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dp \cdot \exp\left\{i\frac{p(x_b-x_a)}{\hbar} - i\frac{D_\alpha|p|^\alpha(t_b-t_a)}{\hbar}\right\}. \quad (44)$$

here $K^{(0)}(x_b t_b|x_a t_a)$ stands for a free particle kernel.

Taking into account Eq.(44) it is easily to check on directly the consistency condition

$$K^{(0)}(x_b t_b|x_a t_a) = \int_{-\infty}^{\infty} dx' K^{(0)}(x_b t_b|x' t') \cdot K^{(0)}(x' t'|x_a t_a).$$

This is a special case of the general quantum-mechanical rule: for events occurring in succession in time the amplitudes are multiplied

$$K(x_b t_b|x_a t_a) = \int_{-\infty}^{\infty} dx' K(x_b t_b|x' t') \cdot K(x' t'|x_a t_a). \quad (45)$$

3.1.1 Fox H-function representation for a free particle kernel

Let's show how a free particle fractional quantum mechanical kernel $K^{(0)}(x_b t_b|x_a t_a)$ defined by Eq.(44) can be expressed in the terms of the Fox H-function [18], [19], [20]. Follow by [17], we obtain the Mellin transform of the quantum mechanical fractional kernel defined by Eq.(44). Comparison of the inverse Mellin transform with the definition of the Fox function yields the desired expression in terms of "known" function, i.e. Fox H-function. Note that H-function bears the name of its discoverer Fox [18] although it have been known at least since 1888, according to [19].

Introducing for simplicity the notations $x \equiv x_b - x_a$, $\tau \equiv t_b - t_a$, we rewrite Eq.(44)

$$K^{(0)}(x,\tau) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dp \cdot \exp\left\{i\frac{px}{\hbar} - i\frac{D_\alpha|p|^\alpha\tau}{\hbar}\right\}. \quad (46)$$

It is easy to see that the relation $K^{(0)}(x,\tau) = K^{(0)}(-x,\tau)$ holds. Hence, it is sufficient to consider $K_L^{(0)}(x,\tau)$ for $x \geq 0$ only. Further, we will use the following definitions of the Mellin transform

$$\hat{K}{}^{(0)}(s,\tau) = \int_0^{\infty} dx x^{s-1} K^{(0)}(x,\tau), \quad (47)$$

and inverse Mellin transform

$$K^{(0)}(x,\tau)=\frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty}dsx^{-s}\hat{K}^{(0)}(s,\tau), \qquad (48)$$

where the integration path is the straight line from $c-i\infty$ to $c+i\infty$ with $0<c<1$.

The Mellin transform of the $K^{(0)}(x,\tau)$ defined in accordance with Eq.(46) is

$$\hat{K}^{(0)}(s,\tau)=\frac{1}{2\pi\hbar}\int\limits_{0}^{\infty}dx\,x^{s-1}\int\limits_{-\infty}^{\infty}dp\cdot\exp\left\{i\frac{px}{\hbar}-i\frac{D_\alpha|p|^\alpha\tau}{\hbar}\right\}.$$

By changing of the variables of integration $p\rightarrow\left(\frac{\hbar}{iD_\alpha\tau}\right)^{1/\alpha}\varsigma$, $x\rightarrow\left(\frac{\hbar}{iD_\alpha\tau}\right)^{1/\alpha}\xi$, one obtains the integrals in the complex ς and ξ planes. Considering the paths of integration in the ς and ξ planes it is easy to represent $\hat{K}^{(0)}(s,\tau)$ as follows

$$\hat{K}^{(0)}(s,\tau)=$$

$$\frac{1}{2\pi}\left(\frac{\hbar}{(\hbar/iD_\alpha\tau)^{1/\alpha}}\right)^{s-1}\int\limits_{0}^{\infty}d\xi\xi^{s-1}\int\limits_{-\infty}^{\infty}d\varsigma\exp\{i\varsigma\xi-|\varsigma|^\alpha\}. \qquad (49)$$

The integrals over $d\xi$ and $d\varsigma$ can be evaluated by using the equation

$$\int\limits_{0}^{\infty}d\xi\xi^{s-1}\int\limits_{0}^{\infty}d\varsigma\exp\{i\varsigma\xi-\varsigma^\alpha\}=\frac{4}{s-1}\sin\frac{\pi(s-1)}{2}\Gamma(s)\Gamma(1-\frac{s-1}{\alpha}), \qquad (50)$$

where $s-1<\alpha\leq 2$ and $\Gamma(s)$ is the gamma function[3].

Inserting Eq.(50) into Eq.(49) and using the functional relations for the gamma function, $\Gamma(1-z)=-z\Gamma(-z)$ and $\Gamma(z)\Gamma(1-z)=\pi/\sin\pi z$, yield

$$\hat{K}^{(0)}(s,\tau)=\frac{1}{\alpha}\left(\frac{\hbar}{(\hbar/iD_\alpha\tau)^{1/\alpha}}\right)^{s-1}\frac{\Gamma(s)\Gamma(\frac{1-s}{\alpha})}{\Gamma(\frac{1-s}{2})\Gamma(\frac{1+s}{2})}.$$

The inverse Mellin transform gives a free particle quantum mechanical kernel $K^{(0)}(x,\tau)$

$$K^{(0)}(x,\tau)=\frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty}dsx^{-s}\hat{K}_L^{(0)}(s,\tau)=$$

[3]The gamma function $\Gamma(s)$ has familiar integral representation $\Gamma(s)=\int\limits_0^\infty dtt^{s-1}e^{-t}$, Re$s>0$.

$$\frac{1}{2\pi i}\frac{1}{\alpha}\cdot\int\limits_{c-i\infty}^{c+i\infty}ds\left(\frac{\hbar}{(\hbar/iD_\alpha\tau)^{1/\alpha}}\right)^{s-1}x^{-s}\frac{\Gamma(s)\Gamma(\frac{1-s}{\alpha})}{\Gamma(\frac{1-s}{2})\Gamma(\frac{1+s}{2})},$$

where the integration path is the straight line from $c-i\infty$ to $c+i\infty$ with $0<c<1$. Replacing s by $-s$ we obtain

$$K^{(0)}(x,\tau)= \tag{51}$$

$$\frac{1}{\alpha}\left(\frac{\hbar}{(\hbar/iD_\alpha\tau)^{1/\alpha}}\right)^{-1}\frac{1}{2\pi i}\int\limits_{-c-i\infty}^{-c+i\infty}ds\left(\frac{1}{\hbar}\left(\frac{\hbar}{iD_\alpha\tau}\right)^{1/\alpha}x\right)^{s}\frac{\Gamma(-s)\Gamma(\frac{1+s}{\alpha})}{\Gamma(\frac{1+s}{2})\Gamma(\frac{1-s}{2})}.$$

The path of integration may be deformed into one running clockwise around R_+-c. Comparison with the definition of the Fox H-function (see, Eqs.(58) and (59), in [17])) leads to

$$K^{(0)}(x,\tau)= \tag{52}$$

$$\frac{1}{\alpha}\left(\frac{\hbar}{(\hbar/iD_\alpha\tau)^{1/\alpha}}\right)^{-1}H^{1,1}_{2,2}\left[\frac{1}{\hbar}\left(\frac{\hbar}{iD_\alpha\tau}\right)^{1/\alpha}x\mid\begin{matrix}(1-1/\alpha,1/\alpha),(1/2,1/2)\\(0,1),(1/2,1/2)\end{matrix}\right].$$

Applying the Property 12.2.5, Ref.[17], of the Fox H-function we can express $K^{(0)}(x,\tau)$ as

$$K^{(0)}(x,\tau)=\frac{1}{\alpha x}H^{1,1}_{2,2}\left[\frac{1}{\hbar}\left(\frac{\hbar}{iD_\alpha\tau}\right)^{1/\alpha}x\mid\begin{matrix}(1,1/\alpha),(1,1/2)\\(1,1),(1,1/2)\end{matrix}\right],\qquad x>0. \tag{53}$$

Or for any x,

$$K^{(0)}(x,\tau)=\frac{1}{\alpha x}H^{1,1}_{2,2}\left[\frac{1}{\hbar}\left(\frac{\hbar}{iD_\alpha\tau}\right)^{1/\alpha}|x|\mid\begin{matrix}(1,1/\alpha),(1,1/2)\\(1,1),(1,1/2)\end{matrix}\right], \tag{54}$$

Substituting $x\equiv x_b-x_a$, $\tau\equiv t_b-t_a$, finally yields

$$K^{(0)}(x_bt_b|x_at_a)=$$

$$\frac{1}{\alpha|x_b-x_a|}H^{1,1}_{2,2}\left[\frac{1}{\hbar}\left(\frac{\hbar}{iD_\alpha(t_b-t_a)}\right)^{1/\alpha}|x_b-x_a|\mid\begin{matrix}(1,1/\alpha),(1,1/2)\\(1,1),(1,1/2)\end{matrix}\right]. \tag{55}$$

This is new equation for 1D free particle fractional quantum mechanical kernel $K^{(0)}(x_bt_b|x_at_a)$.

Let us show that Eq.(55) includes as a particular case at $\alpha = 2$ the well-known Feynman quantum mechanical kernel, see Eq.(3-3) in [1]. Setting in Eq.(55) $\alpha = 2$, applying the series expansion for the function

$$H^{1,1}_{2,2}\left[\frac{1}{\hbar}\left(\frac{\hbar}{iD_\alpha(t_b-t_a)}\right)^{1/2}|x_b-x_a|\ \Big|\ \begin{matrix}(1,1/2),(1,1/2)\\(1,1),(1,1/2)\end{matrix}\right],$$

and finally, substituting $k \to 2l$ yield

$$K^{(0)}(x,\tau)|_{\alpha=2} = \frac{1}{2\hbar}\left(\frac{\hbar}{iD_2\tau}\right)^{1/2}\sum_{l=0}^{\infty}\left(-\frac{1}{\hbar}\left(\frac{\hbar}{iD_2\tau}\right)^{1/2}\right)^{2l}\frac{|x|^{2l}}{(2l)!}\frac{1}{\Gamma(\frac{1}{2}-l)}. \quad (56)$$

Taking into account the identity $\Gamma(\frac{1}{2}+z)\Gamma(\frac{1}{2}-z) = \frac{\pi}{\cos \pi z}$, and applying the Gauss multiplication formula $\Gamma(2l) = \sqrt{\frac{2^{4l-1}}{2\pi}}\Gamma(l)\Gamma(l+\frac{1}{2})$, we find that

$$(2l)!\Gamma(\frac{1}{2}-l) = \frac{\sqrt{\pi}}{(-1)^l}(2)^{2l}l!. \quad (57)$$

With help of Eq.(57) the kernel $K^{(0)}(x,\tau)|_{\alpha=2}$ can be rewritten as

$$K^{(0)}(x,\tau)|_{\alpha=2} = \frac{1}{2\sqrt{\pi}\hbar}\left(\frac{\hbar}{iD_2\tau}\right)^{1/2}\sum_{l=0}^{\infty}\left(-\frac{1}{\hbar}\left(\frac{\hbar}{iD_2\tau}\right)^{1/2}\right)^{2l}\frac{(-1)^l|x|^{2l}}{2^{2l}l!} = \quad (58)$$

$$\frac{1}{2\sqrt{\pi}\hbar}\left(\frac{\hbar}{iD_2\tau}\right)^{1/2}\exp\{-\frac{1}{4}\frac{|x|^2}{\hbar iD_2\tau}\}.$$

Since $D_2 = 1/2m$ we come to the Feynman kernel (see Eq.(3-3), [1])

$$K^{(0)}(x,\tau)|_{\alpha=2} \equiv K^{(0)}_F(x,\tau) = \sqrt{\frac{m}{2\pi i\hbar\tau}}\exp\{\frac{im|x|^2}{2\hbar\tau}\}.$$

Thus, it is shown how Feynman a free particle kernel can be derived from the general equation (55) for the fractional quantum mechanical kernel.

4 Applications of fractional quantum mechanics

4.1 A free particle fractional Schrödinger equation

4.1.1 Scaling properties of 1D fractional Schrödinger equation for a free particle

To make general conclusions regarding solutions of 1D fractional Schrödinger equation for a free particle, let's study the scaling of the solutions. The scale transformations could be written as

$$t = \lambda t', \qquad x = \lambda^{\beta} x', \qquad D_\alpha = \lambda^{\gamma} D'_\alpha, \qquad \psi(x,t) = \lambda^{\delta}\psi(x', t'),$$

where β, γ, δ are exponents of the scale transformations which should leave invariant a free particle 1D fractional Schrödinger equation

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = -D_\alpha (\hbar \nabla)^{\alpha} \psi(x,t), \tag{59}$$

and save the normalization condition $\int\limits_{-\infty}^{\infty} dx |\psi(x,t)|^2 = 1$. It reduces the number of exponents up to 2 and brings the two-parameters scale transformation group

$$t = \lambda t', \qquad x = \lambda^{\beta} x', \qquad D_\alpha = \lambda^{\alpha\beta - 1} D'_\alpha, \qquad \psi(\lambda^{\beta} x, \lambda t) = \lambda^{-\beta/2}\psi(x,t), \tag{60}$$

where β and λ are arbitrary group parameters.

When the initial condition $\psi(x, t = 0)$ is invariant under the scaling group Eq.(60) then the solution of Eq.(59) remains the group invariant. As an example of invariant initial condition one may keep in mind $\psi(x, t = 0) = \delta(x)$, which gives us the Green function of 1D fractional Schrödinger equation.

To get the general scale invariant solutions of 1D fractional Schrödinger equation we may use the renormalization group framework. As far as the scale invariant solutions of Eq.(59) should satisfy the identity Eq.(60) for any arbitrary parameters β and λ, the solutions can depend on combination of x and t to provide the independency of β and λ. Thus, because of existence of the relationships between scaling exponents, $\alpha\beta - \gamma - 1 = 0$ and $\delta + \beta/2 = 0$ the solutions are

$$\psi(x,t) = \frac{1}{x}\Phi(x/(D_\alpha t)^{\frac{1}{\alpha}}) = \frac{1}{(D_\alpha t)^{\frac{1}{\alpha}}}\Psi(x/(D_\alpha t)^{\frac{1}{\alpha}}), \tag{61}$$

where arbitrary functions Φ and Ψ are determined by the conditions, $\Phi(.) = \psi(1,.)$ and $\Psi(.) = \psi(.,1)$.

4.1.2 Exact solution

Following [15], [22] let's solve 1D fractional Schrödinger equation for a free particle (59) with some initial condition $\psi_0(x)$

$$\psi(x, t = 0) = \psi_0(x). \tag{62}$$

Applying the Fourier transforms Eqs.(13) and using the quantum Riesz fractional derivative Eq.(12) yield for the wave function $\varphi(p,t)$ in the momentum representation,

$$i\hbar \frac{\partial \varphi(p,t)}{\partial t} = D_\alpha |p|^{\alpha} \varphi(p,t), \tag{63}$$

with the initial condition $\varphi_0(p)$ given by

$$\varphi_0(p) = \varphi(p, t = 0) = \int\limits_{-\infty}^{\infty} dx e^{-i\frac{px}{\hbar}} \psi_0(x). \tag{64}$$

The solution of the problem Eqs.(63) and (64) is

$$\varphi(p, t) = \exp\{-i\frac{D_\alpha |p|^\alpha t}{\hbar}\}\varphi_0(p), \tag{65}$$

Therefore, the solution of 1D fractional Schrödinger equation Eq.(59) with initial condition given by Eq.(62) can be presented as

$$\psi(x, t) = \frac{1}{2\pi\hbar}\int\limits_{-\infty}^{\infty} dx' \int\limits_{-\infty}^{\infty} dp \exp\{i\frac{p(x-x')}{\hbar} - i\frac{D_\alpha |p|^\alpha t}{\hbar}\}\psi_0(x'), \tag{66}$$

or

$$\psi(x, t) = \tag{67}$$

$$\int\limits_{-\infty}^{\infty} dx' \frac{1}{\alpha(x-x')} H^{1,1}_{2,2}\left[\frac{1}{\hbar}\left(\frac{\hbar}{iD_\alpha t}\right)^{1/\alpha} |x-x'| \mid \begin{array}{c} (1, 1/\alpha), (1, 1/2) \\ (1, 1), (1, 1/2) \end{array}\right] \psi_0(x').$$

Here we expressed the integral over dp in Eq.(66) in terms of $H^{1,1}_{2,2}$-function. If we choose the initial condition $\psi_0(x) = \delta_0(x)$, then Eq.(67) gives us quantum mechanical kernel $K^{(0)}(x, t|0, 0)$ for 1D free particle fractional Schrödinger equation

$$K^{(0)}(x, t|0, 0) = \frac{1}{\alpha x} H^{1,1}_{2,2}\left[\frac{1}{\hbar}\left(\frac{\hbar}{iD_\alpha t}\right)^{1/\alpha} |x| \mid \begin{array}{c} (1, 1/\alpha), (1, 1/2) \\ (1, 1), (1, 1/2) \end{array}\right], \tag{68}$$

or applying the Property 12.2.5 of the Fox H-function (see, [17]) we can write for $K^{(0)}(x, t|0, 0)$

$$K^{(0)}(x, t|0, 0) = \tag{69}$$

$$\frac{1}{\alpha}\left(\frac{\hbar}{(\hbar/iD_\alpha \tau)^{1/\alpha}}\right)^{-1} H^{1,1}_{2,2}\left[\frac{1}{\hbar}\left(\frac{\hbar}{iD_\alpha \tau}\right)^{1/\alpha} |x| \mid \begin{array}{c} (1-1/\alpha, 1/\alpha), (1/2, 1/2) \\ (0, 1), (1/2, 1/2) \end{array}\right].$$

It is easy to see that Eqs.(66) and (69) are scale invariant solutions of 1D fractional Schrödinger equation (see, Eq.(59)) for a free particle.

4.1.3 3D generalization

A free particle quantum dynamics in 3D is governed by following equation (see, Eq.(7))

$$i\hbar \frac{\partial \psi(\mathbf{r},t)}{\partial t} = D_\alpha(-\hbar^2\Delta)^{\alpha/2}\psi(\mathbf{r},t), \qquad \psi(\mathbf{r},t=0)=\psi_0(\mathbf{r}). \tag{70}$$

Using 3D Fourier transformed defined by Eq.(9) and the definition of 3D quantum fractional Riesz derivative given by Eq.(8) yield for the wave function $\varphi(\mathbf{p},t)$ in the momentum representation,

$$i\hbar \frac{\partial \varphi(\mathbf{p},t))}{\partial t} = D_\alpha|\mathbf{p}|^\alpha \varphi(\mathbf{p},t), \tag{71}$$

with the initial condition $\varphi_0(\mathbf{p})$ given by

$$\varphi_0(\mathbf{p}) = \varphi(\mathbf{p},t=0) = \int\limits_{-\infty}^{\infty} d^3r e^{-i\frac{\mathbf{pr}}{\hbar}}\psi_0(\mathbf{r}). \tag{72}$$

Go back to Eq.(70) we can see that the solution $\psi(\mathbf{r},t)$ has a form

$$\psi(\mathbf{r},t) = \frac{1}{(2\pi\hbar)^3}\int\limits_{-\infty}^{\infty} d^3r' \int\limits_{-\infty}^{\infty} d^3p \exp\{i\frac{\mathbf{p}(\mathbf{r}-\mathbf{r}')}{\hbar} - i\frac{D_\alpha|\mathbf{p}|^\alpha t}{\hbar}\}\psi_0(\mathbf{r}').$$

The integral over d^3p can be expressed in terms of $H^{1,2}_{3,3}$-function, see, for instance Eqs.(33) and (34) in.[17]. Thus, the solution of the problem Eq.(70) is

$$\psi(\mathbf{r},t) = \tag{73}$$

$$-\frac{1}{2\pi\alpha}\int\limits_{-\infty}^{\infty} d^3r' \frac{1}{|\mathbf{r}-\mathbf{r}'|^3} H^{1,2}_{3,3}\left[\frac{1}{\hbar}\left(\frac{\hbar}{iD_\alpha t}\right)^{1/\alpha}|\mathbf{r}-\mathbf{r}'|\ \Big|\ \begin{matrix}(1,1),(1,1/\alpha),(1,1/2)\\(1,1),(1,1/2),(2,1)\end{matrix}\right]\psi_0(\mathbf{r}').$$

Substituting into Eq.(73) $\psi_0(\mathbf{r}) = \delta_0(\mathbf{r})$, gives us quantum mechanical kernel $K^{(0)}(\mathbf{r},t|0,0)$ for a free particle 3D fractional Schrödinger equation

$$K^{(0)}(\mathbf{r},t|0,0) = -\frac{1}{2\pi\alpha}\frac{1}{|\mathbf{r}|^3}H^{1,2}_{3,3}\left[\frac{1}{\hbar}\left(\frac{\hbar}{iD_\alpha t}\right)^{1/\alpha}|\mathbf{r}|\ \Big|\ \begin{matrix}(1,1),(1,1/\alpha),(1,1/2)\\(1,1),(1,1/2),(2,1)\end{matrix}\right]. \tag{74}$$

This is new equation for a free particle quantum mechanical 3D kernel. We see that in comparison with 1D case 3D quantum kernel is expressed in the terms of $H^{1,2}_{3,3}$ Fox H-function. In the case $\alpha = 2$ we come to the well-known equation for Feynman 3D quantum kernel $K^{(0)}_F(\mathbf{r},t|0,0)$,

$$K^{(0)}(\mathbf{r},t|0,0)|_{\alpha=2} \equiv K_F^{(0)}(\mathbf{r},t|0,0) = \left(\frac{m}{2\pi i\hbar t}\right)^{3/2} \exp\left\{\frac{im\mathbf{r}|^2}{2\hbar t}\right\}. \tag{75}$$

4.2 The infinite potential well

A particle in a one-dimensional well moves in a potential field $V(x)$ which is zero for $-a \le x \le a$ and which is infinite elsewhere,

$$\begin{aligned} &V(x) = \infty, \qquad x < -a \qquad &\text{(i)} \\ &V(x) = 0, \quad -a \le x \le a \qquad &\text{(ii)} \\ &V(x) = \infty, \qquad x > a \qquad &\text{(iii)} \end{aligned} \tag{76}$$

It is evident *a priori* that the spectrum will be discrete. We are interested in the solutions of the fractional Schrödinger equation (37) that describe the stationary states with well-defined energies. Such a stationary state with an energy E is described by a wave function $\psi(x,t)$ which can be written as $\psi(x,t) = \exp\{-i\frac{Et}{\hbar}\}\phi(x)$,where $\phi(x)$ is now time independent. In the regions (i) and (iii), (see Eq.(76)) we have to substitute ∞ for $V(x)$ into Eq.(37), and it is easily to see that here the fractional Schrödinger equation can be satisfied only if we take $\phi(x) = 0$. In the middle region, (ii), the time-independent fractional Schrödinger equation is

$$-D_\alpha(\hbar\nabla)^\alpha\phi(x) = E\phi(x). \tag{77}$$

We can treat this as a fractional eigenvalue problem [15]. Within region (ii), the eigenfunctions are determined by Eq.(77). Outside of the region (ii), $x < -a$ and $x > a$, the eigenfunctions are zero. We want the wave function $\phi(x)$ to be continuous everywhere, and this means that we impose the boundary conditions $\phi(-a) = \phi(a) = 0$ for the solutions of the fractional differential equation (77). Then the solution of Eq.(77) in the region (ii) can be written as

$$\phi^{\text{even}}(x) = A\cos kx, \qquad \text{or} \qquad \phi^{\text{odd}}(x) = A\sin kx,$$

where the following notation is introduced

$$k = \frac{1}{\hbar}\left(\frac{E}{D_\alpha}\right)^{1/\alpha}, \qquad 1 < \alpha \le 2. \tag{78}$$

The even (under the reflection $x \to -x$) solution $\phi^{\text{even}}(x)$ satisfies the boundary conditions if

$$k = (2m+1)\frac{\pi}{2a}, \qquad m = 0, 1, 2, 3, ...$$

The odd (under the reflection $x \to -x$) solution $\phi^{\mathrm{odd}}(x)$ satisfies the boundary conditions if

$$k = \frac{m\pi}{a}, \qquad m = 1, 2, 3, ... \tag{79}$$

It is easy to check that the normalized solutions are

$$\phi_m^{\mathrm{even}}(x) = \frac{1}{\sqrt{a}} \cos\left\{(m + \frac{1}{2})\frac{\pi x}{a}\right\},$$

and

$$\phi_m^{\mathrm{odd}}(x) = \frac{1}{\sqrt{a}} \sin\frac{m\pi x}{a}.$$

The solutions $\phi^{\mathrm{even}}(x)$ and $\phi^{\mathrm{odd}}(x)$ have the property that

$$\int_{-a}^{a} dx \phi_m^{\mathrm{even}}(x)\phi_n^{\mathrm{even}}(x) = \int_{-a}^{a} dx \phi_m^{\mathrm{odd}}(x)\phi_n^{\mathrm{odd}}(x) = \delta_{mn},$$

$$\int_{-a}^{a} dx \phi_m^{\mathrm{even}}(x)\phi_n^{\mathrm{odd}}(x) = 0,$$

where δ_{mn} is the Kronecker symbol and

The eigenvalues of the particle in a well with help of Eqs.(78) and (79) are [15]

$$E_n = D_\alpha \left(\frac{\pi\hbar}{a}\right)^\alpha n^\alpha, \qquad n = 1, 2, 3...., \qquad 1 < \alpha \leq 2. \tag{80}$$

It is obviously that in the Gaussian case ($\alpha = 2$) Eq.(80) is transformed to the standard quantum mechanical equation (for example, see Eq.(20.7), Ref.[8]) for the energy levels for a particle in a box.

The state of the lowest energy, the ground state, in the infinite potential well is represented by the $\phi_m^{\mathrm{even}}(x)$ at $m = 0$,

$$\phi_{\mathrm{ground}}(x) \equiv \phi_0^{\mathrm{even}}(x) = \frac{1}{\sqrt{a}} \cos\{\frac{\pi x}{2a}\},$$

and its energy is

$$E_{\mathrm{ground}} = D_\alpha \left(\frac{\pi\hbar}{2a}\right)^\alpha. \tag{81}$$

4.3 Fractional Bohr atom

When $V(\mathbf{r})$ is the hydrogenlike potential energy

$$V(\mathbf{r}) = -\frac{Ze^2}{|\mathbf{r}|},$$

where e is the electron charge, Ze is the nuclear charge of the hydrogenlike atom, we come to the eigenvalue problem for fractional hydrogenlike atom,

$$D_\alpha(-\hbar^2\Delta)^{\alpha/2}\phi(\mathbf{r}) - \frac{Ze^2}{|\mathbf{r}|}\phi(\mathbf{r}) = E\phi(\mathbf{r}).$$

This eigenvalue problem had been solved at first in [15]. The total energy is $E = E_{kin} + V$, where E_{kin} is the kinetic energy $E_{kin} = D_\alpha|\mathbf{p}|^\alpha$, and V is the potential energy $V = -\frac{Ze^2}{|\mathbf{r}|}$. It is well-known that if the potential energy is a homogeneous function of the coordinates and the motion takes place in a finite region of space, there exists a simple relation between the time average values of the kinetic and potential energies, known as the *virial theorem* (see, page 23, [27]). It follows from the virial theorem that between average kinetic energy and average potential energy of the system with Hamiltonian (3) there exist the relation

$$\alpha\overline{E}_{kin} = -\overline{V}, \tag{82}$$

where the average value $\overline{f}$ of any function of time is defined as

$$\overline{f} = \lim_{T\to\infty}\frac{1}{T}\int_0^\infty dtf(t).$$

In order to evaluate the energy spectrum of the fractional hydrogenlike atom let us remind the *Niels Bohr postulates* [25],[26]:

1. The electron moves in orbits restricted by the requirement that the angular momentum be an integral multiple of $\hbar$, that is, for circular orbits of radius a_n, the electron momentum is restricted by

$$pa_n = n\hbar, \qquad (n = 1, 2, 3, ...), \tag{83}$$

and furthermore the electrons in these orbits do not radiate in spite of their acceleration. They were said to be in stationary states.

2. Electrons can make discontinuous transitions from one allowed orbit corresponding to $n = n_1$ to another corresponding to $n = n_2$, and the change in energy will appear as radiation with frequency

$$\omega = \frac{E_{n_2} - E_{n_1}}{\hbar}. \tag{84}$$

An atom may absorb radiation by having its electrons make a transition to a higher energy orbit.

Using the first Bohr's postulate and Eq.(82) yields $\alpha D_\alpha \left(\frac{n\hbar}{a_n}\right)^\alpha = \frac{Ze^2}{a_n}$, from which it follows the equation for the radius of the fractional Bohr orbits

$$a_n = a_0 n^{\frac{\alpha}{\alpha-1}}, \tag{85}$$

here a_0 is the fractional Bohr radius (the radius of the lowest, $n = 1$, Bohr orbit) defined as,

$$a_0 = \left(\frac{\alpha D_\alpha \hbar^\alpha}{Ze^2}\right)^{\frac{1}{\alpha-1}}. \tag{86}$$

By using Eq.(82) we find for the total average energy, $\overline{E} = (1-\alpha)\overline{E}_{kin}$. Thus, for the energy levels of the fractional hydrogen-like atom we have

$$E_n = (1-\alpha)E_0 n^{-\frac{\alpha}{\alpha-1}}, \qquad 1 < \alpha \leq 2, \tag{87}$$

where E_0 is the binding energy of the electron in the lowest Bohr orbit, that is, the energy required to put it in a state with $E = 0$ corresponding to $n = \infty$,

$$E_0 = \left(\frac{(Ze^2)^\alpha}{\alpha^\alpha D_\alpha \hbar^\alpha}\right)^{\frac{1}{\alpha-1}}. \tag{88}$$

The energy $(\alpha - 1)E_0$ can be considered as a generalization of the Rydberg constant of standard quantum mechanics. It is easy to see that at $\alpha = 2$ the energy $(\alpha-1)E_0$ is transformed into the well-known expression for the Rydberg constant, $\mathrm{Ry} = me^4/2\hbar^2$.

The frequency of the radiation ω associated with the transition, say, for example from m to n, $m \to n$, is, according to the second Bohr postulate,

$$\omega = \frac{(1-\alpha)E_0}{\hbar} \cdot \left[\frac{1}{n^{\frac{\alpha}{\alpha-1}}} - \frac{1}{m^{\frac{\alpha}{\alpha-1}}}\right]. \tag{89}$$

The new equations (85)-(89) bring us fractional generalization of the "Bohr atom" theory. In the special Gaussian case (standard quantum mechanics) Eqs.(85)-(89) allow us to reproduce the well-known results of the Bohr theory [25], [26].

4.4 Fractional oscillator

4.4.1 Quarkonium and fractional oscillator

As an another physical application of the developed FQM we propose a new fractional approach to study the quark-antiquark $q\overline{q}$ bound states treated within the non-relativistic potential picture [2]. Note, that only for heavy quark systems (for example, charmonium $c\overline{c}$ or bottonium $b\overline{b}$) the non-relativistic approach can be justified. The term quarkonium is used to denote any $q\overline{q}$ bound state system [28] in analogy to positronium in the e^+e^- system. The non-relativistic potential approach remains the most successful and simplest way to calculate and predict energy levels and decay rates.

Thus, from stand point of "potential" view, we can assume that the confining potential energy of two quarks localized say, at the space points $\mathbf{r}_i$ and $\mathbf{r}_j$ is given by

$$V(|\mathbf{r}_i - \mathbf{r}_j|) = q_i q_j |\mathbf{r}_i - \mathbf{r}_j|^\beta, \tag{90}$$

where q_i and q_j are the color charges of i and j quarks respectively and the index $\beta > 0$. Equation.(90) coincides with the QCD requirements: (i) At short distances the quarks and gluons appear to be weakly coupled; (ii) At large distances the effective coupling becomes strong, resulting in the phenomena of quark confinement[4].

Considering the N-quarks statistical system yields the following equation for the potential energy $U(\mathbf{r}_1, ..., \mathbf{r}_N)$ of the system,

$$U(\mathbf{r}_1, ..., \mathbf{r}_N) = \sum_{1 \le i < j \le N} q_i q_j |\mathbf{r}_i - \mathbf{r}_j|^\beta. \tag{91}$$

In order to illustrate the main idea, we consider the simplest case, when color q_i charge only can be q or $-q$ and the colorless condition $\sum\limits_{i=1}^{N} q_i = 0$ takes place. Using the general statistical mechanics approach (see the Definition 3.2.1 and the Proposition 3.2.2, Ref. [21]) we can conclude that only for $0 < \beta \le 2$ the many particle system with the potential energy (91) will be thermodynamically stable.

In order to study the problem of quarkonium it seems reasonable to consider the non-relativistic FQM model with the fractional Hamiltonian operator $H_{\alpha,\beta}$ defined as

$$H_{\alpha,\beta} = D_\alpha(-\hbar^2\Delta)^{\alpha/2} + q^2|\mathbf{r}|^\beta, \quad 1 < \alpha \le 2, \quad 1 < \beta \le 2, \tag{92}$$

where $\mathbf{r}$ is 3D vector, $\Delta = \partial^2/\partial\mathbf{r}^2$ is the Laplacian, and the operator $(-\hbar^2\Delta)^{\alpha/2}$ is defined by Eq.(8).

For the special case, when $\alpha = \beta$ the Hamiltonian operator (92) has a form

$$H_\alpha = D_\alpha(-\hbar^2\Delta)^{\alpha/2} + q^2|\mathbf{r}|^\alpha, \quad 1 < \alpha \le 2. \tag{93}$$

It is easy to see that the Hamiltonian H_α is the fractional generalization of 3D harmonic oscillator Hamiltonian of standard quantum mechanics. Follow [2], [16], we will call by the fractional oscillators the quantum mechanical models with the Hamiltonians given by the Eqs.(92) and (93).

It would be interesting to calculate the energies of the bound states and the decay rates based on the FQM models with the Hamiltonians (92) and (93). These results could be compared with the experimental statistical data of J/ψ decays. The information on decay rates and angular distribution provides an ideal testing ground for the fractional models of $q\overline{q}$ bound states.

[4]The term quark confinement describes the observation that quarks do not occur isolated in nature, but only in hadronic bound states as the colorless objects such as baryons and mesons.

4.4.2 Spectrum of 1D fractional oscillator in semiclassical approximation

The 1D fractional oscillator with the Hamilton function $H_\alpha = D_\alpha |p|^\alpha + q^2|x|^\beta$ poses an interesting problem for semiclassical treatment [16]. We set the total energy equal to E, so that

$$E = D_\alpha |p|^\alpha + q^2|x|^\beta, \tag{94}$$

whence $|p| = \left(\frac{1}{D_\alpha}(E - q^2|x|^\beta)\right)^{1/\alpha}$. At the turning points $p = 0$. Thus, classical motion is thus possible in the range $|x| \leq (E/q^2)^{1/\beta}$.

A routine use of the Bohr-Sommerfeld quantization rule [8] yields

$$2\pi\hbar(n + \frac{1}{2}) = \oint pdx = 4\int\limits_0^{x_m} pdx = 4\int\limits_0^{x_m} D_\alpha^{-1/\alpha}(E - q^2|x|^\beta)^{1\alpha}dx, \tag{95}$$

where the notation $\oint$ means the integral over one complete period of the classical motion, $x_m = (E/q^2)^{1/\beta}$ is the turning point of classical motion. To evaluate the integral in the right hand of Eq.(95) we introduce a new variable $y = x(E/q^2)^{-1/\beta}$. Then we have

$$\int\limits_0^{x_m} D_\alpha^{-1/\alpha}(E - q^2|x|^\beta)^{1\alpha}dx = \frac{1}{D_\alpha^{1/\alpha}q^{2/\beta}}E^{\frac{1}{\alpha}+\frac{1}{\beta}}\int\limits_0^1 dy(1 - y^\beta)^{1/\alpha}.$$

The integral over dy can be expressed in the terms of the B-function. Indeed, substitution $z = y^\beta$ yields[5]

$$\int\limits_0^1 dy(1 - y^\beta)^{1/\alpha} = \frac{1}{\beta}\int\limits_0^1 dzz^{\frac{1}{\beta}-1}(1 - z)^{\frac{1}{\alpha}} = \frac{1}{\beta}B(\frac{1}{\beta}, \frac{1}{\alpha} + 1). \tag{97}$$

With help of Eq.(97) we rewrite Eq.(95) as

$$2\pi\hbar(n + \frac{1}{2}) = \frac{4}{D_\alpha^{1/\alpha}q^{2/\beta}}E^{\frac{1}{\alpha}+\frac{1}{\beta}}\frac{1}{\beta}B(\frac{1}{\beta}, \frac{1}{\alpha} + 1).$$

The above equation gives the values of the energy of stationary states for 1D fractional oscillator [16],

[5]The B-function is defined by

$$B(u, v) = \int\limits_0^1 dyy^{u-1}(1 - y)^{v-1}. \tag{96}$$

$$E_n = \left(\frac{\pi\hbar\beta D_\alpha^{1/\alpha} q^{2/\beta}}{2B(\frac{1}{\beta}, \frac{1}{\alpha}+1)} \right)^{\frac{\alpha\beta}{\alpha+\beta}} \cdot (n+\frac{1}{2})^{\frac{\alpha\beta}{\alpha+\beta}}. \tag{98}$$

This equation is generalized the well-known energy spectrum of the standard quantum mechanical oscillator (see for example, [8]) and is transformed to it at $\alpha = 2$, $\beta = 2$.

We note that at

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1, \tag{99}$$

Eq.(98) gives the equidistant energy spectrum. When $1 < \alpha \leq 2$ and $1 < \beta \leq 2$ the condition given by Eq.(99) takes place for $\alpha = 2$ and $\beta = 2$ only. It means that only standard quantum mechanical oscillator has equidistant energy spectrum.

5 Some solvable models of fractional quantum mechanics

Let's outlook a few analytically solvable problems of fractional quantum mechanics recently elaborated in [29].

5.1 Bound state in δ–potential well

For one dimensional attractive δ–potential well, $V(x) = -\gamma\delta(x)$, $(\gamma > 0)$, where $\delta(x)$ is the Dirac delta function, fractional Schrödinger equation Eq.(37) becomes

$$-D_\alpha(\hbar\nabla)^\alpha \phi(x) - \gamma\delta(x)\phi(x) = E\phi(x), \qquad 1 < \alpha \leq 2. \tag{100}$$

In any one-dimensional attractive potential there will be a bound state, that is $E < 0$. In this case Dong and Xu [29], found the energy and the wave function of the bound state. The bound energy has a form

$$E = -\left(\frac{\gamma B(1/\alpha, 1-1/\alpha)}{\pi\hbar\alpha D_\alpha^{1/\alpha}} \right)^{\alpha/(\alpha-1)}, \qquad 1 < \alpha \leq 2, \tag{101}$$

here $B(1/\alpha, 1-1/\alpha)$ is the B-function defined by Eq.(96).

The wave function $\phi(x)$ of the bound state is

$$\phi(x) = -\frac{\gamma C}{2\pi\hbar^2 E\alpha|x|} H_{2,3}^{2,1} \left[|x| \left(-\frac{D_\alpha\hbar^\alpha}{E} \right)^{-1/\alpha} \Big| \begin{array}{c} (1-\frac{1}{\alpha}, \frac{1}{\alpha}), (\frac{1}{2}, \frac{1}{2}) \\ (1,1), (1-\frac{1}{\alpha}, \frac{1}{\alpha}), (\frac{1}{2}, \frac{1}{2}) \end{array} \right], \tag{102}$$

here the constant $C = \phi(0)$ such, that normalization condition

$$\int_{-\infty}^{\infty} dx|\phi(x)|^2 = 1, \tag{103}$$

has to be satisfied.

5.2 Linear potential field

Considering a particle in a linear potential field (for example, see [8], page 74), the potential function $V(x)$ can be written as:

$$V(x) = \{ \begin{matrix} Fx & x \geq 0, (F > 0) \\ \infty & x < 0, \end{matrix} \tag{104}$$

and fractional Schrödinger equation Eq.(37) becomes

$$-D_\alpha(\hbar\nabla)^\alpha\phi(x) + Fx\phi(x) = E\phi(x), \qquad 1 < \alpha \leq 2, \qquad x \geq 0. \tag{105}$$

The continuity and bounded conditions of the wave function, let us conclude that $\phi(x) = 0,\ x < 0$. Besides, $\phi(x)$ must satisfy the boundary conditions

$$\begin{matrix} \phi(x) = 0, & x = 0, \\ \phi(x) = 0, & x \to \infty. \end{matrix} \tag{106}$$

Then, wave function $\phi_n(x)$ of the quantum state with energy E_n, $n = 1, 2, 3, ...$is [29]

$$\phi_n(x) = \tag{107}$$

$$\frac{2\pi A}{\alpha+1} H_{2,2}^{1,1}\left[(x - \frac{E_n}{F})\frac{1}{\hbar}\left(\frac{D_\alpha}{(\alpha+1)F\hbar}\right)^{-\frac{1}{\alpha+1}} \mid \begin{matrix} (1-\frac{1}{\alpha+1}, \frac{1}{\alpha+1}), (\frac{\alpha+2}{2(\alpha+1)}, \frac{\alpha}{2(\alpha+1)}) \\ (0,1), (\frac{\alpha+2}{2(\alpha+1)}, \frac{\alpha}{2(\alpha+1)}) \end{matrix} \right], \tag{108}$$

with the constant A given by

$$A = \frac{1}{2\pi\hbar}\left(\frac{D_\alpha}{(\alpha+1)F\hbar}\right)^{-1/(\alpha+1)}, \tag{109}$$

and the energy spectra E_n

$$E_n = \lambda_n F\hbar\left(\frac{D_\alpha}{(\alpha+1)F\hbar}\right)^{-1/(\alpha+1)}, \qquad 1 < \alpha \leq 2, \qquad n = 1, 2, 3, ..., \tag{110}$$

where λ_n are the solutions of the equation [29]

$$H^{1,1}_{2,2}\left[-\lambda_n \mid \begin{array}{c}(1-\frac{1}{\alpha+1},\frac{1}{\alpha+1}),(\frac{\alpha+2}{2(\alpha+1)},\frac{\alpha}{2(\alpha+1)})\\ (0,1),(\frac{\alpha+2}{2(\alpha+1)},\frac{\alpha}{2(\alpha+1)})\end{array}\right]=0. \tag{111}$$

When $\alpha = 2$ Eqs.(108) and (110) turn into well-known equations of standard quantum mechanics [8], [29].

Other solvable physical models of fractional quantum mechanics include 1D Coulomb potential [29], a finite square potential well, dynamics in the field of 1D lattice, penetration through a $\delta-$potential barrier, the Dirac comb [30], the bound state problem and penetration through double $\delta-$potential barrier [31].

6 Fractional statistical mechanics

6.1 Density matrix

In order to develop the fractional statistical mechanics (FSM) let us go in Eq.(40) from imaginary time to "inverse temperature" $\beta = 1/k_BT$, where k_B is the Boltzmann's constant and T is the temperature, $it \to \hbar\beta$. Then the partition function Z is expressed as a trace of the density matrix $\rho_L(x, \beta|x_0)$ [2], [14]

$$Z = \int dx \rho_L(x, \beta|x) =$$

$$\int dx \int\limits_{x(0)=x(\beta)=x} \mathrm{D}x(\tau) \int \mathrm{D}p(\tau) \exp\{-\frac{1}{\hbar}\int\limits_0^{\hbar\beta} du \left\{-ip(u)\dot{x}(u) + H_\alpha(p(u), x(u)\right\}, \tag{112}$$

where the fractional Hamiltonian $H_\alpha(p, x)$ has form (38) and $p(u), x(u)$ may be considered as paths running along on "imaginary time axis", $u = it$. The exponential expression of Eq.(112) is very similar to the fractional canonical action (42). Since it governs the fractional quantum-statistical path integrals it may be called the fractional quantum-statistical action or fractional Euclidean action, indicated by the superscript (e),

$$S^{(\mathrm{e})}_\alpha(p, x) = \int\limits_0^{\hbar\beta} du\{-ip(u)\dot{x}(u) + H_\alpha(p(u), x(u)\}. \tag{113}$$

The parameter u is not the true time in any sense. It is just a parameter in an expression for the density matrix (see, for instance, [1]). Let us call u the "time", leaving the quotation marks to remind us that it is not real time (although u does have the dimension of time). Likewise $x(u)$ will be called the "coordinate" and $p(u)$ the "momentum". Then Eq.(112) may be interpreted in following way: Consider all the possible paths by which the system can travel between the initial $x(0)$ and final $x(\beta)$ configurations in the "time" $\hbar\beta$.

The fractional density matrix ρ_L is a path integral over all possible paths, the contribution from a particular path being the "time" integral of the canonical action (113) (considered as the functional of the path $p(u), x(u)$ in the phase space) divided by $\hbar$. The partition function is derived by integrating over only those paths for which initial $x(0)$ and final $x(\beta)$ configurations are the same and after that we integrate over all possible initial (or final) configurations.

6.1.1 A free particle

The fractional density matrix $\rho_L^{(0)}(x, \beta|x_0)$ of a free particle ($V = 0$) can be written as [2], [14]

$$\rho_L^{(0)}(x, \beta|x_0) = \frac{1}{2\pi\hbar}\int\limits_{-\infty}^{\infty} dp \exp\left\{i\frac{p(x-x_0)}{\hbar} - \beta D_\alpha |p|^\alpha\right\} = \tag{114}$$

$$= \frac{1}{\alpha|x-x_0|} H_{2,2}^{1,1}\left[\frac{|x-x_0|}{\hbar(D_\alpha\beta)^{1/\alpha}} \mid \begin{array}{c}(1,1/\alpha),(1,1/2)\\(1,1),(1,1/2)\end{array}\right],$$

where $H_{2,2}^{1,1}$ is the Fox H-function (see, [18]-[20]).
For 1D system of space scale Ω the trace of Eq.(114) reads

$$Z = \int\limits_\Omega dx \rho_L^{(0)}(x, \beta|x_0) = \frac{\Omega}{2\pi\hbar}\frac{1}{(\beta D_\alpha)^{1/\alpha}}\Gamma(\frac{1}{\alpha}).$$

When $\alpha = 2$ and $D_2 = 1/2m$ Eq.(114) gives the well-known density matrix for 1D free particle (see Eq.(10-46) of Ref. [1] or Eq.(2-61) of Ref. [32])

$$\rho^{(0)}(x, \beta|x_0) = \left(\frac{m}{2\pi\hbar^2\beta}\right)^{1/2} \exp\left\{-\frac{m}{2\hbar^2\beta}(x-x_0)^2\right\}. \tag{115}$$

The Fourier representation $\rho_L^{(0)}(p, \beta|p')$ of the fractional density matrix $\rho_L^{(0)}(x, \beta|x_0)$ defined by

$$\rho_L^{(0)}(p, \beta|p') = \int\limits_{-\infty}^{\infty} dx dx_0 \rho_L^{(0)}(x, \beta|x_0) \exp\{-\frac{i}{\hbar}(px - p'x_0)\}$$

can be rewritten as

$$\rho_L^{(0)}(p, \beta|p') = 2\pi\hbar\delta(p-p') \cdot e^{-\beta D_\alpha |p|^\alpha}.$$

In order to obtain a formula for the fractional partition function in the limit of fractional classical mechanics let us study the case when $\hbar\beta$ is small. It is easy to see that the fractional density matrix $\rho_L(x, \beta|x_0)$ can be written as

$$\rho_L(x, \beta|x_0) = e^{-\beta V(x_0)}\frac{1}{2\pi\hbar}\int\limits_{-\infty}^{\infty} dp \exp\left\{i\frac{p(x-x_0)}{\hbar} - \beta D_\alpha |p|^\alpha\right\}.$$

Then the partition function Z in the limit of classical mechanics becomes

$$Z = \int_{-\infty}^{\infty} dx \rho_L(x, \beta|x) = \frac{\Gamma(1/\alpha)}{2\pi\hbar(\beta D_\alpha)^{1/\alpha}} \int_{-\infty}^{\infty} dx e^{-\beta V(x)}, \tag{116}$$

where $\Gamma(1/\alpha)$ is the gamma function.

The partition function Z given by Eq.(116) is an approximation valid if the particles of the system cannot wander very far from their initial positions in the "time" $\hbar\beta$. The limit on the distance which the particles can wander before the approximation breaks down can be estimated from Eq.(114). We see that if the final point differs from the initial point by as mush as

$$\Delta x \simeq \hbar(\beta D_\alpha)^{1/\alpha} = \hbar \left(\frac{D_\alpha}{kT}\right)^{1/\alpha}$$

the exponential function of Eq.(114) becomes greatly reduced. From this, we can infer that intermediate points only on paths which do not contribute greatly to the path integral of Eq.(114). Thus, we conclude that if the potential $V(x)$ does not alter very much as x moves over this distance, then the fractional classical statistical mechanics is valid.

6.1.2 Motion equation for the density matrix

The density matrix $\rho_L(x, \beta|x_0)$ obeys the fractional differential equation [2], [14]

$$-\frac{\partial \rho_L(x, \beta|x_0)}{\partial \beta} = -D_\alpha(\hbar\nabla_x)^\alpha \rho_L(x, \beta|x_0) + V(x)\rho_L(x, \beta|x_0) \tag{117}$$

or

$$-\frac{\partial \rho_L(x, \beta|x_0)}{\partial \beta} = H_\alpha \rho_L(x, \beta|x_0), \quad \rho_L(x, 0|x_0) = \delta(x - x_0), \tag{118}$$

where the fractional Hamiltonian H_α is defined by Eq.(15). The last equation can be considered as fractional generalization of the Bloch equation for density matrix [33].

6.1.3 3D generalization of FSM

The above developments can be generalized to 3D dimension. It is obviously that a free particle density matrix $\rho_L^{(0)}(\mathbf{r}, \beta|\mathbf{r}_0)$ for 3D case has a form

$$\rho_L^{(0)}(\mathbf{r}, \beta|\mathbf{r}_0) = \frac{1}{(2\pi\hbar)^3} \int d^3p \cdot \exp\left\{ i\frac{\mathbf{p}(\mathbf{r} - \mathbf{r}_0)}{\hbar} - \beta D_\alpha |\mathbf{p}|^\alpha \right\}, \tag{119}$$

where $\mathbf{r}$, $\mathbf{r}_0$ and $\mathbf{p}$ are 3D vectors.

To present the density matrix $\rho_L(\mathbf{r},\beta|\mathbf{r}_0)$ in the terms of the Fox H-function we rewrite Eq.(119) as

$$\rho_L^{(0)}(\mathbf{r},\beta|\mathbf{r}_0) = \frac{1}{2\pi^2\hbar^2|\mathbf{r}-\mathbf{r}_0|}\int\limits_0^\infty dpp\sin(\frac{p|\mathbf{r}_b-\mathbf{r}_a|}{\hbar})\exp\{-\beta D_\alpha|\mathbf{p}|^\alpha\}\,.$$

With help of the identity $\rho_L^{(0)}(\mathbf{r},\beta|\mathbf{r}_0) = -\frac{1}{2\pi}\frac{\partial}{\partial x}\rho_L^{(0)}(x,\beta|0)|_{x=|\mathbf{r}-\mathbf{r}_0|}$,where $\rho_L^{(0)}(x,\beta|0)$ is 1D density matrix given by Eq.(114), we find

$$\rho_L^{(0)}(\mathbf{r},\beta|\mathbf{r}_0) = -\frac{1}{2\pi\alpha}\frac{1}{|\mathbf{r}-\mathbf{r}_0|^3}H_{3,3}^{1,2}\left[\frac{|\mathbf{r}-\mathbf{r}_0|}{\hbar(D_\alpha\beta)^{1/\alpha}}\mid\begin{array}{c}(1,1),(1,1/\alpha),(1,1/2)\\(1,1),(1,1/2),(2,1)\end{array}\right]. \tag{120}$$

This is new equation for a free particle fractional density matrix in 3D space.

The density matrix $\rho_L(\mathbf{r},\beta|\mathbf{r}_0)$ obeys the fractional differential equation

$$-\frac{\partial\rho_L(\mathbf{r},\beta|\mathbf{r}_0)}{\partial\beta} = D_\alpha(-\hbar^2\Delta)^{\alpha/2}\rho_L(\mathbf{r},\beta|\mathbf{r}_0) + V(\mathbf{r})\rho_L(\mathbf{r},\beta|\mathbf{r}_0), \tag{121}$$

or

$$-\frac{\partial\rho_L(\mathbf{r},\beta|\mathbf{r}_0)}{\partial\beta} = H_\alpha\rho_L(\mathbf{r},\beta|\mathbf{r}_0), \qquad \rho_L(\mathbf{r},\beta=0|\mathbf{r}_0) = \delta(\mathbf{r}-\mathbf{r}_0), \tag{122}$$

where 3D fractional Hamiltonian H_α is defined by Eq.(3).

Thus, the Eqs. (114), (117)-(122) are fundamental equations of fractional statistical mechanics.

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